I.

THE ORIGIN OF THE PLANETS.

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WITH FOUR PLATES.

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THE ORIGIN OF THE PLANETS.

1. If the several members of our solar system be considered with regard to their distance from the Sun, a very curious relation will be found to exist between them. It is this:

> The major axes of their orbits are such that the mean motion of each body is an almost exact multiple of its neighbor's in one of the three proportions $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$, and in the case of Venus and the Earth $\frac{3}{5}$; these ratios corresponding to closely adjacent places in space.

Investigation into the cause of such commensurability reveals an important law:

Each planet has formed the next in order, of necessity at exactly one of these commensurable points, at the same time displacing it slightly sunward. In other words, the positions of the planets are not haphazard but have been determined *seriatim* each by its predecessor; thus showing that the solar system is an articulated whole, an inorganic organism, which not only evolved but evolved in a definite order, the steps of which celestial mechanics enable us to retrace.

This I shall now proceed to show.

2. The orbital elements of any planet are:

$$\begin{aligned}
m &= \text{the mass},\\
a &= \text{the major axis},\\
n &= \text{the mean motion},\\
e &= \text{the excentricity},\\
\varpi &= \text{the longitude of the perihelion},\\
I &= \text{the inclination to the ecliptic},\\
\Omega &= \text{the ascending node},\\
\epsilon &= \text{the mean longitude at the epoch}.
\end{aligned}$$

Unaccented these will refer to the inner planet, accented to the outer.

$$M = \text{the mass of the Sun, which will be taken as unity.}$$

$$M + m = \mu.$$

In elliptic motion $n^2a^3 = \mathrm{f}\mu$, where f is the constant of gravitation.

The potential of the perturbing force, called the perturbing function, is for the outer planet on the inner

$$R = \mathrm{f}\frac{m}{\rho} - \mathrm{f}\frac{m'r\sigma}{r'^2},$$

and for the inner on the outer

$$R = \mathrm{f}\frac{m}{\rho} - \mathrm{f}\frac{mr'\sigma}{r^2}.$$

To find the motion from this expression recourse has to be had to series, first in powers of the masses and then in those of the excentricities and inclinations, which, because the masses of the planets are very small compared with the Sun and the excentricities and inclinations for the principal planets are also small, prove rapidly convergent, if the time considered be not too long. Expressed in ascending powers of the latter, the first term $\mathrm{f}\frac{m'}{\rho}$ gives

$$\begin{aligned} R_1 = {} & \Sigma\,[\tfrac{1}{2}A^{(i)} - \tfrac{1}{2}\eta^2 B^{(i-1)} + \frac{e^2+e'^2}{8}(-4i^2A^{(i)} + 2A_1^{(i)} + 2A_2^{(i)})]\cos i\,(l'-\lambda) \\ & + \tfrac{1}{2}e\Sigma\,[-2iA^{(i)} - A_1^{(i)}]\cos[il' - (i-1)\lambda - \omega] \\ & + \tfrac{1}{2}e'\Sigma\,[(2i+1)A^{(i)} + A_1^{(i)}]\cos[(i+1)l' - i\lambda - \varpi'] \\ & + \tfrac{1}{8}e^2\Sigma\,[(4i^2-5i)A^{(i)} + (4i-2)A_1^{(i)} + 2A_2^{(i)}]\cos[il' - (i-2)\lambda - 2\omega], \text{ etc., etc.} \end{aligned}$$

In which

$$A^{(i)} = \frac{f_{\frac{1}{2}}^{(i)}(\alpha)}{a'}, \qquad B^{(i)} = \frac{f_{\frac{3}{2}}^{(i)}(\alpha)}{a'}, \quad \text{and} \quad A_n^{(i)} = \frac{a^n}{1,2\ldots n}\frac{d^nA^{(i)}}{da^n}, \qquad B_n^{(i)} = \frac{a^n}{1,2\ldots n}\frac{d^nB^{(i)}}{da^n},$$

where

$$\alpha = \frac{a}{a'}.$$

The second part $\frac{-\mathrm{f}m'r\sigma}{r'^2}$, for an outer planet on an inner, merely adds to R in $A^{(1)}$ and $A_1^{(1)}$ $-\frac{a}{a'^2}$; in $B^{(0)}$ and $B_1^{(0)}$, $-\frac{2a}{a'^2}$; while $-\mathrm{f}\frac{mr'\sigma}{r^2}$, for an inner on an outer, adds in $A^{(1)}$, $-\frac{a'}{a^2}$, in $A_n^{(1)}$, $(-1)^{n+1}(n+1)\frac{a'}{a^2}$, in $B^{(0)}$, $-\frac{2a'}{a^2}$, and in $B_n^{(0)}$, $(-1)^{n+1}2(n+1)\frac{a'}{a^2}$.

As we shall treat the planets as lying in one plane, consequently

$$\lambda = l + \tau' - \tau \text{ becomes } l = nt + \epsilon, \text{ the mean longitude; and } \omega = \varpi + \tau' - \tau,$$

where τ and τ' are the angular distances to the intersection of the two orbits reckoned first along the reference plane and thence along the orbits, becomes ϖ.

3. Analytic processes give us the changes in the elements with regard to the time in well-known equations, of which those we shall have occasion to use are, for m' on m:

$$\frac{da}{dt} = \frac{2m'na^2}{\mu}\frac{dR_1}{d\epsilon},$$

$$\frac{dn}{dt} = -\frac{3m'n^2a}{\mu}\frac{dR_1}{d\epsilon},$$

$$\frac{d\epsilon}{dt} = -\frac{2m'na^2}{\mu}\frac{dR_1}{da} + \frac{m'na\sqrt{1-e^2}}{\mu e}(1-\sqrt{1-e^2})\frac{dR_1}{de} + \frac{m'na\tan\frac{i}{2}}{\mu\sqrt{1-e^2}}\frac{dR_1}{di}.$$

Inspection of R will show that to the first order in the masses neither a nor n have any secular terms, terms, that is, increasing with the time; since $\frac{da}{dt}$ and $\frac{dn}{dt}$ both depend upon $\frac{dR}{d\epsilon}$ which enters only into the periodic ones. Not so $\frac{d\epsilon}{dt}$. For $\frac{d\epsilon}{dt}$ has constant terms introduced by $\frac{dR}{da}$, $\frac{dR}{de}$ and $\frac{dR}{di}$. When, therefore, it is integrated, giving $\delta\epsilon$, a term st is produced, which combines with nt to change permanently the mean motion. Analogically we get $\delta\epsilon'$, and $\delta\epsilon$ and $\delta\epsilon'$ are always of opposite signs.

Thus for Uranus and Neptune we find, taking the important term only:

$$\frac{d\epsilon}{dt} = -\frac{m'}{\mu}naf(a) \qquad \text{and} \qquad \frac{d\epsilon'}{dt} = +\frac{m}{\mu'}n'f'(a),$$

whence

$$\delta\epsilon = -.467\frac{m'}{\mu}nt \qquad \text{and} \qquad \delta\epsilon' = +3.001\frac{m}{\mu'}n't.$$

The geometric explanation of this is that the outer planet acting in opposition to the sun, amounts to diminishing the sun's mass, thus decreasing the mean motion at the same distance. The inner planet's action is equivalent to an increase of the Sun's mass which increases the mean motion. Thus a body already formed tends to draw surrounding matter to itself by making that matter's mean motion more nearly synchronous with its own.

4. But another factor, less conspicuous, is even more concerned in such attraction. As regards the periodic terms R may be written

$$R = \sum \mathrm{f}m'\frac{f(a)}{a'}\left(\frac{e}{2}\right)^h\left(\frac{e'}{2}\right)^{h'}\cos\left\{(i'n'-in)t + i'\epsilon' - i\epsilon - k\varpi - k'\varpi'\right\}$$

in which by the mode of development:

i' and i are always whole numbers;

$h + h' +$ an even number, zero included, $= i' - i$;

$h = k +$ an even number, zero included;

and $h' = k' +$ an even number, zero included,

Consider any one of these terms in

$$\frac{da}{dt} = \frac{2\,nm'a^2}{\mu}\frac{dR_1}{d\epsilon}.$$

$$\frac{dR_1}{d\epsilon} = \frac{f(\alpha)}{a'}\left(\frac{e}{2}\right)^h\left(\frac{e'}{2}\right)^{h'} i\cdot\sin\left\{(i'n' - in)\,t + i'\epsilon' - i\epsilon - k\varpi - k'\varpi'\right\},$$

which, calling the angle θ, gives

$$\frac{da}{dt} = +\,2\frac{m'}{\mu}\,na\,\alpha\,f(\alpha)\left(\frac{e}{2}\right)^h\left(\frac{e'}{2}\right)^{h'} i\cdot\sin\theta,$$

and integrating,

$$\delta a = -\,2\frac{m'}{\mu}\,na\alpha\,f(\alpha)\left(\frac{e}{2}\right)^h\left(\frac{e'}{2}\right)^{h'}\frac{i}{i'n' - in}\cos\theta.$$

5. Usually these terms have small coefficients and large arguments because $i'n' - in$ is generally a good-sized quantity.

But in certain cases the reverse is true. If

$$i' : i :: n : n',$$

or nearly so, then $i'n' - in = 0$, or nearly so, and the coefficient of δa becomes ∞ or very great.

Such terms, then, surpass enormously the others and instantly dominate δa in the series.

Now the relation $$\frac{i'}{i} = \frac{n}{n'}$$

means that the mean motions are commensurable, inasmuch as i and i' always are whole numbers.

a, therefore, could not remain at the point of commensurability of period, since the periods are inversely as the mean motions, nor even in the immediate neighborhood of the point. For $i'n' - in$ continues large for some distance on either side of it.

6. For what happens consider the equation

$$\delta a = -\,2\frac{m'}{\mu}\left(\frac{e}{2}\right)^h\left(\frac{e'}{2}\right)^{h'} na\alpha\,f(\alpha)\cdot\frac{i}{i'n' - in}\cos\theta,$$

in which

$$\theta = (i'n' - in)\,t + i'\epsilon' - i\epsilon - k\varpi - k'\varpi'.$$

So long as n does not increase till $i'n' - in = 0$ for any value of θ, there is an oscillation of a, timed to θ, in which a never crosses the commensurable point. The farther away a is from this point for a given initial value of θ, the smaller δa and the less the swing, 1. because $i'n' - in$ is larger and the coefficient smaller; 2. because the period of θ is less.

7. When $i'n' - in$ approaches zero for general values of θ the action alters. Tisserand has discussed it in the case of the asteroids. There are three possible cases. Putting for simplicity for $i'n' - in$ the particular value $2n' - n_1$ we have:

1. $$(2n' - n_1)^2 > 2m'h_1^2 \cos^2\frac{\theta_1}{2};$$

that is, when the constant $c^2 > 1$ or θ makes complete revolutions.

2. $$(2n' - n_1)^2 < 2m'h_1^2 \cos^2\frac{\theta_1}{2};$$

that is, when $c^2 < 1$ or θ librates.

3. $$(2n' - n_1)^2 = 2m'h_1^2 \cos^2\frac{\theta_1}{2},$$

the dividing value of the two in all of which

$$h_1^2 = 3\frac{n'^2}{a^2}e\frac{f(a)}{a'},$$

and the subscripts denote the values at the start.

He found that the extreme values of n were in the several cases: approx.

1. $$\left\{2n' - \frac{n_1}{2} + \frac{1}{2}\sqrt{(n_1 - 2n')^2 - 2m'h_1^2\cos^2\frac{\theta_1}{2}}\right\}\frac{n_1}{n'} < n,$$

and $$n < \frac{n_1}{n'}\left\{2n' - \frac{n_1}{2} + \frac{1}{2}\sqrt{(n_1 - 2n')^2 + 2m'h_2'\sin^2\frac{\theta_1}{2}}\right\}.$$

2. $$\left\{2n' - \frac{n_1}{2} - \frac{1}{2}\sqrt{(n_1 - 2n')^2 + 2m'h_1^2\sin^2\frac{\theta_1}{2}}\right\}\frac{n_1}{n'} < n,$$

and $$n < \frac{n_1}{n'}\left\{2n' - \frac{n_1}{2} + \frac{1}{2}\sqrt{(n_1 - 2n')^2 + 2m'h_1^2\sin^2\frac{\theta_1}{2}}\right\}.$$

3. $$n_1 \quad \text{and} \quad \frac{n_1^2}{2n'}.$$

In case 1. the particle never crosses the commensurable point; in 2. it librates over it; in 3. if the particle were at the point to start with, it would never leave it.

In case 1. the particle lingers relatively long in the neighborhood of the commensurable point, being then at the end of its swing. In case 2. the particle has a long swing, greater than in the case 1. as the equations show, since its outer limit is the same approx. while its inner is much farther in. Case 3. is a mere conceptual dividing line between the other two.

8. Tisserand's investigation takes no account of Jupiter's orbital excentricity, nor of higher terms, i. e., 2θ, etc., nor of $\frac{de}{dt}$ or $\frac{d\varpi}{dt}$. Nevertheless it sufficiently illustrates the swing in the several cases for our present purpose. More accurately considered by

another method I have shown elsewhere that among the asteroids around gaps $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$ and $\frac{2}{3}$ there are no instances of case 2,[1] no asteroids, that is, that exhibit libration.

The important thing in our present investigation is the amplitude of the swing. This it will be noted increases as the commensurable point is approached. There is thus an excursionary freedom near the points which is lacking elsewhere.

9. Since e is always less than unity the lower its power the greater δa and δn. The most effective terms, therefore, are those in which $i' - i = 1$, such as $\frac{1}{2}$, $\frac{2}{3}$ . . . $\frac{100}{101}$ and so on. The next most effective where $i' - i = 2$, such as $\frac{1}{3}$, $\frac{2}{4}$, $\frac{3}{5}$. These are followed by those of $i' - i = 3$, as $\frac{2}{5}$.

Each forms a series the members of which cluster closer together the nearer they approach their originating planet.

They also become more effective at close quarters. For the expression for δa is multiplied by i, which increases with approach. $f(\alpha)$ also increases as $\alpha = \frac{a}{a'}$ approaches unity.

Any difference of density in a revolving nebula is thus a starting point for accumulation. So soon as two or three particles have gathered together there they tend by increased mass to annex their neighbors. An embryo planet is thus formed. By the same principle it grows crescendo through an ever increasing sphere of influence until the commensurable points are too far apart to bridge by their oscillation the space between them.

10. This is the chief factor that has enabled the planets to sweep their surroundings clear. Nor is it evident that a ring of matter could not fold back into a localized mass, for though there are critical points at about 60° longitude from the Sun corresponding to Lagrange's equilateral triangle solution, speed and direction of motion must there be particularly adjusted to secure stability. The fact that four and only four asteroids survive to exemplify Lagrange's special case of stable motion is in itself evidence of the necessary rarity of the occurrence. It is certainly interesting that only with Jupiter are such outliers known. For, as we shall see later, (§11 and §21, 1, 2, 3) to suppose the planets generally to have started as collected masses is incompatible with their present positions; and though Jupiter alone of the major ones, may have done so, it is more analogic to suppose not.

11. Beyond a certain distance from the planet the commensurate-period swings no longer suffice to bridge the intervening space and the planet's annexing power stops. This happens somewhat before a certain place is reached where three potent periodic ratios succeed each other — $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$. For here the distances between the periodic points is greatly

[1] See Phil. Mag., March, 1912, and Astr. Journal, n° 630. The gap $\frac{2}{3}$ has been worked out by the writer, but not published.

increased, as Diagram I shows. The effect is well exemplified by the surviving asteroids indicated by the rough plot of their major axes in Diagram II.

At this distance a new action sets in. Though the character of its occasioning be the same it produces a very different outcome. The greater swing of the particles at these commensurate points together with a temporary massing of some of them near it conduces to collisions and near approaches between them which must end in a certain permanent combining there. A nucleus of consolidation is thus formed. This attracts other particles to it, gaining force by what it feeds on, until out of the once diffused mass a new planet comes into being which in its turn gathers to itself the matter about it.

A new planet tends to collect here: because the annexing power of the old has here ceased while at the same time the scattered constituents to compose it are here aided to combine by the very potent commensurability perturbations of its already formed neighbor.

So soon as it has come into being another begins to be beyond it, called up in the same manner. It could not do so earlier because the most important *deus ex machina* in the matter, the perturbation of its predecessor, was lacking.

So the process goes on, each planet acting as a sort of elder sister in bringing up the next.

That such must have been the genesis of the several planets is evident when we consider that had each arisen of itself out of surrounding matter there would have been in celestial mechanics nothing to prevent their being situated in almost any relative positions other than the peculiar one in which they actually stand.

That all the planets are ordered where this reasoning would place them a glance at the Diagrams III and IV will show.

12. We now come to an important corroborative bit of telltale evidence. It will be noticed that the several planets are not quite at the commensurate points. They are in fact all just inside them, as will appear more conspicuously in the sequel. Now this offing turns out to be an inevitable consequence of the same cause that tended to mass them originally at the commensurate points. It thus affords visible confirmation of the former's truth. This we shall now show.

So far we have considered only the commensurability terms which depend on the first power of the masses. We now proceed to terms of the second order in them.

To get the second order terms the elements entering the first must themselves be deemed variable. As the differentials of the elements include the mass, those to the second order introduce by substitution the second order of the masses. We shall denominate the first order increments by $\delta_1 a$ etc.; the second order ones by $\delta_2 a$.

Taking both into account,

$$a \text{ becomes } a + \delta_1 a + \delta_2 a,$$

and

$$n \quad \text{“} \quad n + \delta_1 n + \delta_2 n.$$

Introduce these into the equation of an elliptic orbit

$$n^2 a^3 = \mathrm{f}\mu,$$

and it becomes, where $\mu = 1 + m$ and f is the constant of gravitation,

$$(n + \delta_1 n + \delta_2 n)^2 (a + \delta_1 a + \delta_2 a)^3 = n^2 a^3 = \mathrm{f}\mu.$$

Expanding and equating like powers of m' we get:

$$\delta_1 n = -\frac{3\,n}{2\,a}\,\delta_1 a,$$

$$\delta_2 n = -\frac{3\,n}{2\,a}\,\delta_2 a + \frac{15}{8}\frac{n}{a^2}(\delta_1 a)^2.$$

$\int ndt$ therefore becomes

$$ndt - \frac{3\,n}{2\,a}\int \delta_1 a\,dt - \frac{3\,n}{2\,a}\int d_2 a\,dt + \frac{15\,n}{8\,a^2}\int (\delta_1 a)^2\,dt.$$

Now, there are no secular terms in a or n to the first order of the masses, as Laplace and Lagrange showed. Nor are there any such to the second order, as Poisson proved. But consider the third term in the perturbation.

Since
$$\delta_1 a = -2\frac{m'}{\mu}\,na\alpha f(\alpha)\left(\frac{e}{2}\right)^h\left(\frac{e'}{2}\right)^{h'}\frac{i}{i'n' - in}\cos\theta,$$

$$(\delta_1 a)^2 = 4\left(\frac{m'}{\mu}\right)^2 n^2 a^2 \alpha^2\,[f(\alpha)]^2\left(\frac{e}{2}\right)^{2h}\left(\frac{e'}{2}\right)^{2h'}\frac{i^2}{(i'n' - in)^2}\cos^2\theta.$$

Now
$$\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta.$$

Consequently

$$\int (\delta_1 a)^2\,dt = 2\left(\frac{m'}{\mu}\right)^2 n^2 a^2 \alpha^2\,[f(\alpha)]^2\left(\frac{e}{2}\right)^{2h}\left(\frac{e'}{2}\right)^{2h'}\frac{i^2}{(i'n'-in)^2}\,t + \text{a periodic term} = s_2 t + \text{a periodic term}.$$

Here then we have a permanent addition made to the mean motion, nt becoming $nt + s_2 t = (n + s_2)\,t$ and since $(\delta_1 a)^2$ is a square, s_2 will always be positive. This of course applies to all the periodic terms in $\delta_1 a$ but the only important one is that near the commensurable point, since $i'n' - in$ is there very small.

13. Suppose now a particle or planet close to the commensurable point inside it. The mean motion in consequence of the above perturbation will be permanently increased,

and therefore the major axis be permanently decreased. In other words the particle or planet will be pushed sunward. If it be still where $i'n' - in$ be small it will suffer another push, and so on until it has reached a place where the perturbation is no longer sensible.

Suppose the particle outside the point at the place where the perturbation begins to be sensible. As $(\delta_1 a)^2$ is necessarily positive whatever the sign of $\delta_1 a$ it will be pushed in, the action becoming more violent till the body reaches the commensurable point over which it will go to join in its turn those previously displaced. Thus no particles or planets can remain in this swept swarth, which on both sides braids the point.

14. We thus reach from theory two conclusions:

1. All the planets were originally forced to form where the important and closely lying commensurable points $\frac{1}{2}$, $\frac{2}{5}$, or $\frac{1}{3}$, and in one case $\frac{3}{5}$, existed with their neighbors; which of these points it was being determined by the perturbations themselves.

2. Each planet was at the same time pushed somewhat sunward by perturbation due to the same cause but in the second order of the masses.

15. We shall now see by turning to the planets themselves how dutifully their major axes fall into line with these formative effects of theory. To do this we shall begin by calculating the change in the major axes for the commensurate period terms in $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$ and $\frac{3}{5}$ of the several mean motions due the important terms in the first order in the masses. Those to the second are, for the secular terms, their squares. Actually the results here given are for the present positions of the planets, but the deductions to be drawn from them are the same as if they were for the commensurate points.

Perturbations of the Major Axis.

Terms of Long Period.

Uranus and Neptune.

$$\frac{1}{2}\left\{\begin{array}{lll}
\delta a & = & 4.947\, m' a\, \alpha \dfrac{e}{2} \dfrac{n}{2n' - n} \cos\{(2n' - n)t + 2\epsilon' - \epsilon - \varpi\} \dfrac{1}{\mu} \\[2ex]
\delta a' & = & -7.907\, m a'\, \alpha \dfrac{e}{2} \quad `` \quad `` \quad `` \quad `` \quad `` \quad \dfrac{1}{\mu'} \\[2ex]
\delta a & = & -1.813\, m' a\, \alpha \dfrac{e'}{2} \dfrac{n}{2n' - n} \cos\{(2n' - n)t + 2\epsilon' - \epsilon - \varpi'\} \dfrac{1}{\mu} \\[2ex]
\delta a' & = & 3.226\, m a'\, \alpha \dfrac{e'}{2} \quad `` \quad `` \quad `` \quad `` \quad `` \quad \dfrac{1}{\mu'}
\end{array}\right.$$

Uranus and Saturn.

$$
\tfrac{1}{3}\left\{
\begin{array}{lll}
\delta a &= -5.387\, m'a\, \alpha \left(\frac{e}{2}\right)^2 \frac{n}{3n'-n} \cos\{(3n'-n)\,t + 3\epsilon' - \epsilon - 2\varpi\} & \frac{1}{\mu} \\[2ex]
\delta a' &= 11.396\, m\, a'\, \alpha \left(\frac{e}{2}\right)^2 \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'} \\[2ex]
\delta a &= 19.256\, m'a\, \alpha\, \frac{ee'}{4}\, \frac{n}{3n'-n} \cos\{(3n'-n)\,t + 3\epsilon' - \epsilon - \varpi - \varpi'\} & \frac{1}{\mu} \\[2ex]
\delta a' &= -40.737\, m\, a'\, \alpha\, \frac{ee'}{4} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'} \\[2ex]
\delta a &= -3.290\, m'a\, \alpha \left(\frac{e'}{2}\right)^2 \frac{n}{3n'-n} \cos\{(3n'-n)\,t + 3\epsilon' - \epsilon - 2\varpi'\} & \frac{1}{\mu} \\[2ex]
\delta a' &= 9.701\, m\, a'\alpha \left(\frac{e'}{2}\right)^2 \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'}
\end{array}\right.
$$

Saturn and Jupiter.

$$
\tfrac{2}{5}\left\{
\begin{array}{lll}
\delta a &= 37.23\, m'a\, \alpha \left(\frac{e}{2}\right)^3 \frac{n}{5n'-2n} \cos\{(5n'-2n)\,t + 5\epsilon' - 2\epsilon - 3\varpi\} & \frac{1}{\mu} \\[2ex]
\delta a' &= -68.74\, m\, a'\alpha \left(\frac{e}{2}\right)^3 \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'} \\[2ex]
\delta a &= -186.0\, m'a\, \alpha \left(\frac{e}{2}\right)^2 \frac{e'}{2}\, \frac{n}{5n'-2n} \cos\{(5n'-2n)t + 5\epsilon' - 2\epsilon - 2\varpi - \varpi'\} & \frac{1}{\mu} \\[2ex]
\delta a' &= 343.4\, m\, a'\, \alpha \left(\frac{e}{2}\right)^2 \frac{e'}{2} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'} \\[2ex]
\delta a &= 307.7\, m'a\, \alpha \left(\frac{e}{2}\right)\left(\frac{e'}{2}\right)^2 \frac{n}{5n'-2n} \cos\{(5n'-2n)\,t + 5\epsilon' - 2\epsilon - \varpi - 2\varpi'\} & \frac{1}{\mu} \\[2ex]
\delta a' &= -568.1\, m\, a'\, \alpha \left(\frac{e}{2}\right)\left(\frac{e'}{2}\right)^2 \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'} \\[2ex]
\delta a &= -167.9\, m'a\, \alpha \left(\frac{e'}{2}\right)^3 \frac{n}{5n'-2n} \cos\{(5n'-2n)\,t + 5\epsilon' - 2\epsilon - 3\varpi'\} & \frac{1}{\mu} \\[2ex]
\delta a' &= 302.9\, m\, a'\, \alpha \left(\frac{e'}{2}\right)^3 \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'}
\end{array}\right.
$$

Venus and the Earth.

$$
\tfrac{3}{5}\left\{
\begin{array}{lll}
\delta a &= -87.56\, m'a\, \alpha \left(\frac{e}{2}\right)^2 \frac{n}{5n'-3n} \cos\{(5n'-3n)\,t + 5\epsilon' - 3\epsilon - 2\varpi\} & \frac{1}{\mu} \\[2ex]
\delta a' &= 124.11\, m\, a'\alpha \left(\frac{e}{2}\right)^2 \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'} \\[2ex]
\delta a &= 227.81\, m'a\, \alpha\, \frac{e}{2}\, \frac{e'}{2}\, \frac{n}{5n'-3n} \cos\{(5n'-3n)\,t + 5\epsilon' - 3\epsilon - \varpi - \varpi'\} & \frac{1}{\mu} \\[2ex]
\delta a' &= -322.92\, m\, a'\alpha\, \frac{e}{2}\, \frac{e'}{2} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'} \\[2ex]
\delta a &= -147.22\, m'a\, \alpha \left(\frac{e'}{2}\right)^2 \frac{n}{5n'-3n} \cos\{(5n'-3n)\,t + 5\epsilon' - 3\epsilon - 2\varpi'\} & \frac{1}{\mu} \\[2ex]
\delta a' &= 208.68\, m\, a'\alpha \left(\frac{e'}{2}\right)^2 \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{''} & \frac{1}{\mu'}
\end{array}\right.
$$

From them we note that:

1. The inner planet is *caeteris paribus* more potent than the outer.

2. The greater the mass of the disturber and, in certain cases, the greater the excentricity of either the disturber or the disturbed the greater the effect.

16. In most cases the effect of each component of the pair is somewhat masked at first by the simultaneous action of the other. But fortunately there is a set of instances in which the effect of one is sensibly eliminated, allowing that of the other to show unhid. This is the case with Jupiter and the asteroids. So small are the masses of the latter, taken either singly or together, that they produce no perceptible effect on Jupiter, while, on the other hand, Jupiter's giant mass enables us to mark his action as through a magnifying glass, minutely and alone.

17. Diagram V gives the plot of the major axes of all the asteroids except those that have nothing to do with the argument: the four at Jupiter's distance, Achilles, Nestor, Hector and Patroclus; Thule just inside the $\frac{3}{4}$ point; and at the other end Hungaria and Eros.

We notice;

1. The gathering of the great bulk of the asteroids around the commensurate periods $\frac{1}{2}$, $\frac{2}{5}$, and $\frac{1}{3}$.

2. The swaths cut by the terms in m'^2 at the points, nearly symmetric on either side of them.

3. Their width corresponding to the importance of the terms; that is, to the distances from the commensurable points at which the effect becomes insensible or counteracted by other causes.

4. The asteroidal condensation at their inner edges, the embryos of planets destined in this case never to be born.

5. The fact that had the asteroids been sufficiently numerous they must have collected at the $\frac{2}{5}$ point and gathered in the outliers there.

No laboratory experiment could more prettily exemplify a law than does this asteroid witness in the sky.

18. Turning now to the outer planets and remembering their respective masses and eccentricities,

	Mass.	Ratio to the next in order.	Excentricity.
Jupiter	$\frac{1}{1047.35}$	3.4	.04836
Saturn	$\frac{1}{3502}$	6.4	.05584
Uranus	$\frac{1}{22760}$	.84	.04708
Neptune	$\frac{1}{19000}$		.00854

and that the inner are, other things equal, three or four times as effective from position as the outer of each pair, we see that in all cases we ought to have just what we observe: the inner outside, the outer inside the commensurate points, since what we now perceive is the difference of the sunward shoves of the two.

19. Next the inner or terrestrial planets. The Earth and Mars repeat the story of the outer ones at $\frac{1}{2}$, Venus and the Earth do the same at $\frac{3}{5}$.[2] That this should be $\frac{3}{5}$ instead of $\frac{1}{2}$ is apparently to be ascribed to the more nearly equal perturbation of the two for points between. For though Venus' mass is less, her inner position makes up for it, giving her a slight mastery over the debatable territory.

Thus for Venus: and for the Earth-Moon

$$m = .805 \qquad m' = 1.$$
$$a = .723 \qquad a' = 1.$$
$$e = .0068 \qquad e' = .0167$$

and

$$\frac{\delta a}{\delta a'} = \frac{\text{coef. } m'a}{\text{coef. } ma'} = \frac{7}{11}.$$

Mutually regarded Venus' pull therefore outdoes the Earth's. So that here again the action is as for the outer planets. Lastly, in the case of Mercury and Venus, Mercury's mass, $\frac{1}{23}$ of the Earth, being only $\frac{1}{18}$ of Venus', is so small that position cannot make up for it and Venus thrusts Mercury sunward more than she herself is pulled in the same direction, the period being $\frac{2}{5}$.

20. In explanation of why the particular commensurate relations exist in the several cases the following may be offered. When the action of the more potent planet greatly exceeds the other's it sweeps to itself particles farther away than would otherwise be possible. As the ratio of the two pulls increases toward equality the eminent domain of the stronger shrinks, allowing the other to form nearer. Thus the commensurate point passes in from $\frac{1}{3}$ to $\frac{2}{5}$, $\frac{1}{2}$ and finally $\frac{3}{5}$. Now the ratio of the two pulls is the nearest unity between Venus and the Earth; less between Uranus and Neptune, still less between Jupiter and Saturn, and less still between Saturn and Uranus, as calculation from Table I will show. And this is the order of their commensurate points: $\frac{3}{5}$, $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$. The masses themselves which brought this about must be ascribed to initial differing densities of the originally scattered nebula. Between Mercury and Venus the ratio is $\frac{2}{5}$, as it should be. Between Jupiter and the asteroids $\frac{2}{5}$ instead of $\frac{1}{3}$, because the second order terms have pushed them from him instead of pulling them toward him had they lain outside. For the Earth and Mars we should have expected $\frac{1}{3}$ instead of $\frac{1}{2}$, but this may be due to the

2 $\frac{8}{13}$ is now nearer, but was a much less potent period in the past.

continued action of the gigantic Jupiter in this territory, or it may be that a second origin of condensation started with the Earth while Jupiter fashioned the outer planets.

21. From the foregoing some interesting deductions are possible:

1. The planets grew out of scattered material. For had they arisen from already more or less complete nuclei these could not have borne to one another the general commensurate relation of mean motions existent today.

2. Each brought the next one into being by the perturbation it induced in the scattered material at a definite distance from it.

3. Jupiter was the starting point, certainly as regards the major planets; and is the only one among them that could have had a nucleus at the start, though that, too, may equally have been lacking.

4. After this was formed Saturn, then Uranus, and then Neptune.

This order is curiously supported by the present mean densities of the four. Their densities and masses are:—

	Density ($\oplus$ = 1).	Mass.
Jupiter	.24	$\frac{1}{1047.35}$
Saturn	.13	$\frac{1}{3502}$
Uranus	.22	$\frac{1}{22760}$
Neptune	.20	$\frac{1}{19000}$

Density is determined by:

1. Original mass;
2. Age.

Now the younger a planet and the greater the mass, the longer must it take to cool and condense. If, then, Neptune had led off it should have now much the greatest density, followed by Uranus, and so on. If, however, Jupiter started first, his mass would to a certain extent offset his age, Saturn might be what he is, while Uranus and Neptune being smaller might already have caught up in the race for condensation, thus equalizing them with Jupiter, which we observe to be the case.

5. The asteroids point unmistakably to such a genesis, missed in the making.

6. The inner planets betray *inter se* the action of the same law, and dovetail into the major ones through the $\frac{2}{5}$ relation again between Mars and the asteroids.

We thus close with the law we enunciated: *Each planet has formed the next in the series at one of the adjacent commensurable-period points, corresponding to* $\frac{1}{2}$, $\frac{2}{5}$, $\frac{1}{3}$, *and in one instance* $\frac{3}{5}$, *of its mean motion, each then displacing the other slightly sunward, thus making of the solar system an articulated whole, an inorganic organism, which not only evolved but evolved in a definite order, the steps of which celestial mechanics enables us to retrace.*

The above planetary law may perhaps be likened to Mendelief's law for the elements. It, too, admits of prediction. Thus in conclusion I venture to forecast that when the nearest trans-Neptunian planet is detected it will be found to have a major axis of very approximately 47.5 astronomical units, and from its position a mass comparable with that of Neptune, though probably less; while, if it follows a feature of the satellite systems which I have pointed out elsewhere, its excentricity should be considerable, with an inclination to match.

April 14, 1913.

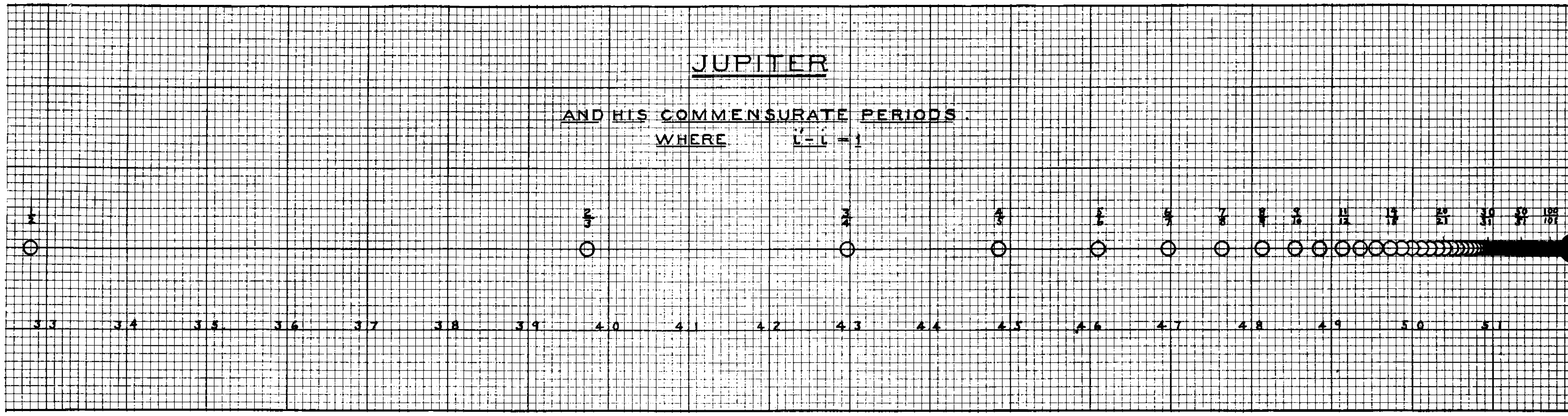

Diagram I.

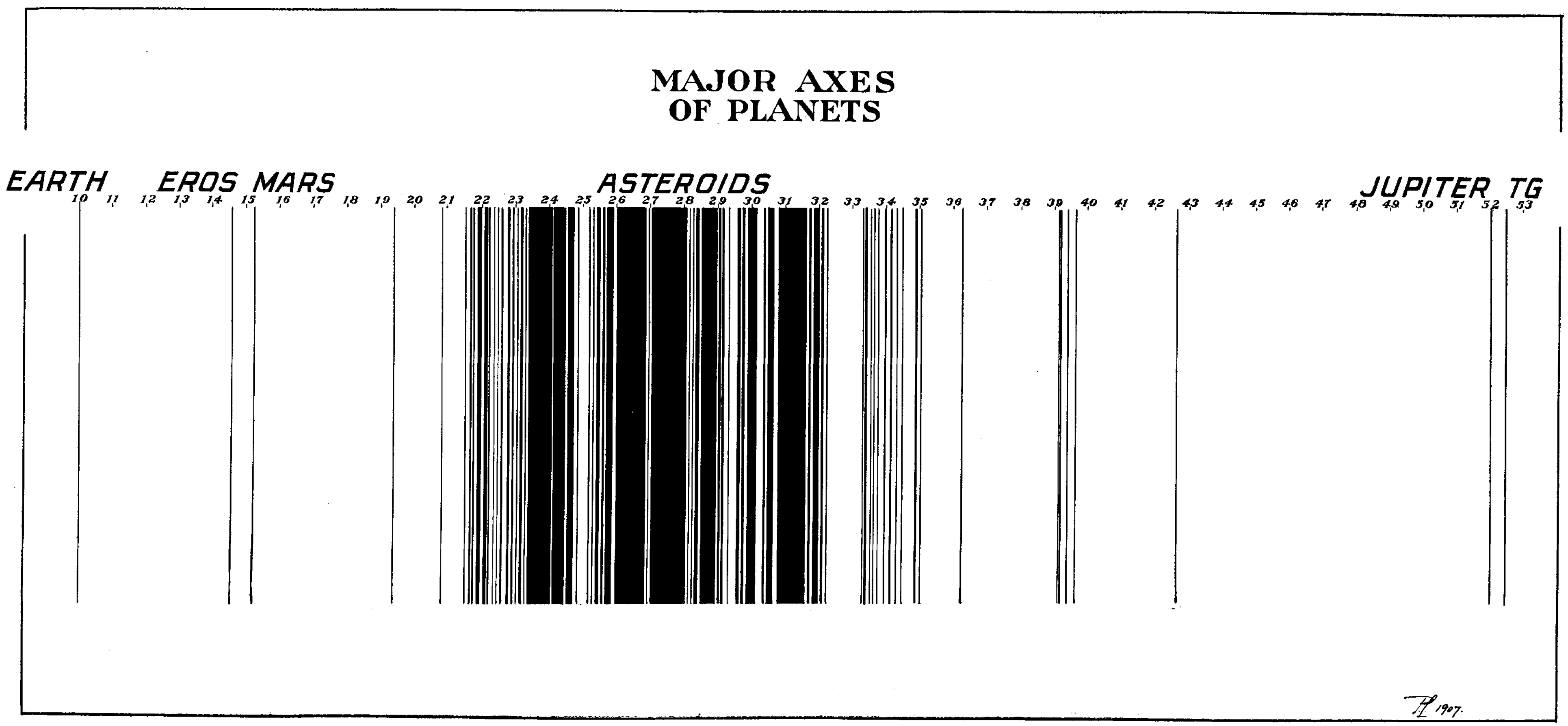

Diagram II.

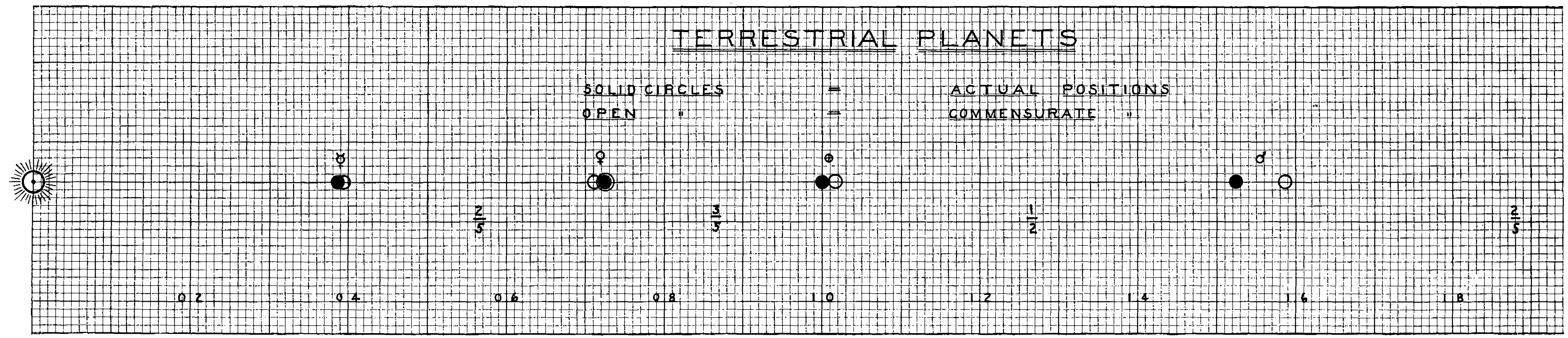

DIAGRAM III.

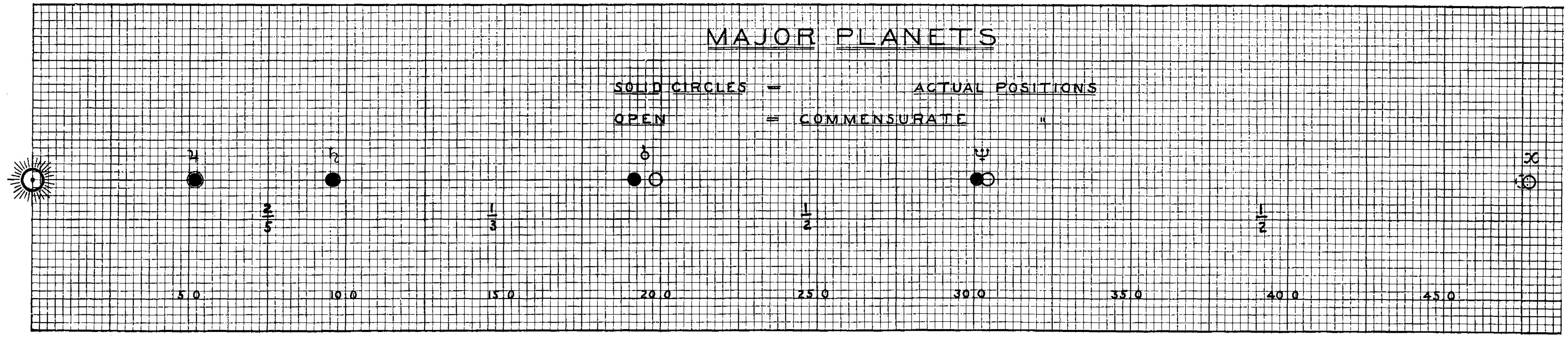

DIAGRAM IV.

LOWELL.—ORIGIN OF PLANETS.

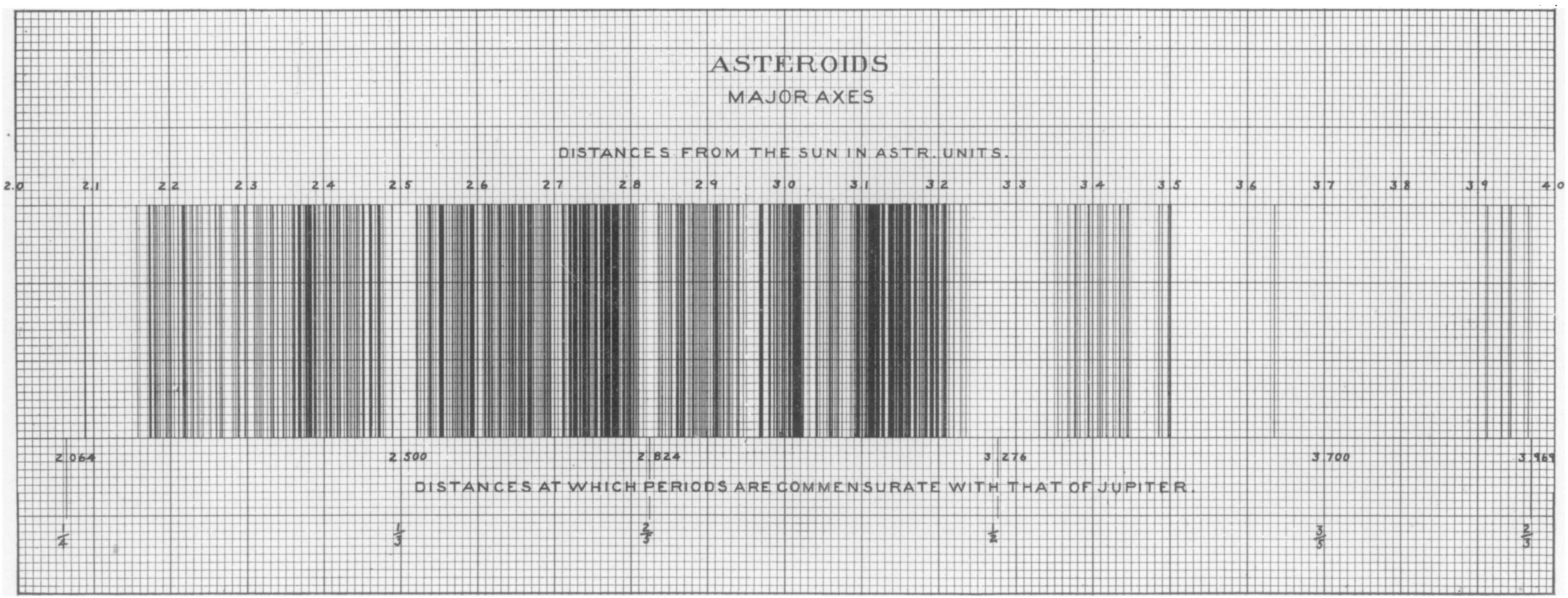

DIAGRAM V.

LOWELL. — ORIGIN OF THE PLANETS.

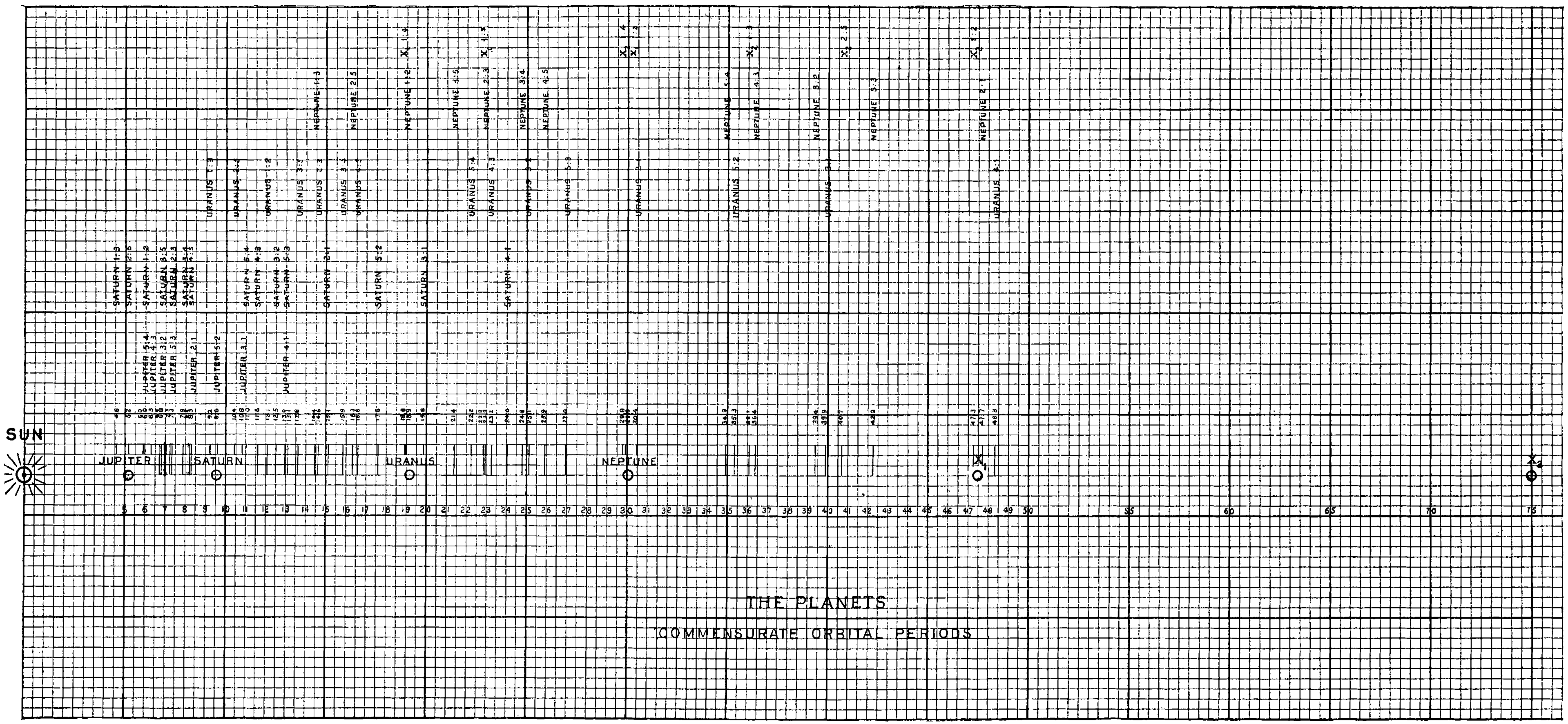

DIAGRAM VI.

LOWELL.—ORIGIN OF PLANETS.